B.SC. FOURTH SEMESTER (HONOURS) EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 42113**

Course Code: SH/MTH/403/C-10 **Course Title: Ring Theory and Linear Algebra-I**

Full Marks: 40 **Time: 2 Hours**

<u>The figures in the margin indicate full marks</u>

Notations and symbols have their usual meaning.

1. **Answer *any five* from the following questions:** $\mathbf{2\times 5=10}$

a) Find all the solutions of $x^3 = x$ in the ring $(\mathbb{Z}_6, +, \cdot)$.

b) Give an example of a right ideal in a ring which is not a left ideal.

c) Give an example of a ring homomorphism which is not an isomorphism.

d) Find the kernel of the homomorphism $f: \mathbb{Z} \to \mathbb{Z}_6$ defined by $f(n) = [n]$ for all $n \in \mathbb{Z}$.

e) Let V be the vector space of real valued continuous functions over $\mathbb{R}$, then show that the set W of solutions of $3\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0$ is a subspace of V.

f) Give an example of a set A of vectors in the vector space $\mathbb{R}^3$ over $\mathbb{R}$ such that A is a spanning set for the vector space but not linearly independent.

g) Consider the following map $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (\sin x_1, x_2)$. Is f a linear operator on $\mathbb{R}^2$ over $\mathbb{R}$? Justify your answer.

h) Consider the following linear operator $T: \mathbb{R}^2 \to \mathbb{R}^2$ $(over\ \mathbb{R})$ defined by $T(x_1, x_2) = (-x_1, x_1)$. Find the matrix representation of T with respect to the standard ordered basis $B = \{(1,0), (0,1)\}$.

2. **Answer *any four* from the following questions:** $\mathbf{5\times 4=20}$

a) Examine whether the following sets are field or not.
(i) $\{a + b\sqrt{2}: a, b \in \mathbb{Z}\}$ (ii) $\{a + b\sqrt{2}: a, b \in \mathbb{Q}\}$. 5

b) State and prove second isomorphism theorem for rings. $1 + 4 = 5$

c) Let R be a commutative ring with 1 ($1 \neq 0$). Then prove that a proper ideal P of R is prime if and only if the quotient ring R/P is an integral domain. 5

d) Show that the set $B = \{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1,0,-1), \alpha_2 = (1,2,1), \alpha_3 = (0,-3,2)$ forms a basis for the vector space $\mathbb{R}^3$ over $\mathbb{R}$. Express each of the standard ordered basis vectors as linear combination of $\alpha_1, \alpha_2, \alpha_3$. $3 + 2 = 5$

e) Define subspace of a vector space. Let V be a vector space over the field F. Prove that intersection of any two subspaces of V is again a subspace of V. Consider the vector space $M_n(\mathbb{R})$ of all real $n \times n$ matrices over $\mathbb{R}$. Is $W = \{A \in M_n(\mathbb{R}) : A \text{ is invertible}\}$ a subspace of $M_n(\mathbb{R})$? Justify your answer. $\mathbf{1 + 2 + 2 = 5}$

f) Let T be a linear operator on $\mathbb{R}^3$ over $\mathbb{R}$ such that the matrix representation of T with respect to the standard ordered basis $B = \{(1,0,0), (0,1,0), (0,0,,1)\}$ is $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$. Find a basis for (i) the null space of T and (ii) the range space of T. $\mathbf{2}$ **½** $+$ $\mathbf{2}$ **½** $\mathbf{= 5}$

3. Answer *any one* from the following questions: $\mathbf{10 \times 1 = 10}$

a) (i) Define maximal ideal with an example.

(ii) Define field of quotients of an integral domain D.

(iii) Let V be a vector space over the field of real numbers and $\{\alpha, \beta, \gamma\} \subseteq V$be a set of linearly independent vectors. Then prove $(\alpha + \beta), (\beta + \gamma), (\gamma + \alpha)$ are also linearly independent.

(iv) Let F be a field and consider the vector space $F^2 = \{(a, b) : a, b \in F\}$ over F. Find the T^{-1}, if exists, where T is linear operator defined on F^2 by $T(x_1, x_2) = (x_1 + x_2, x_1)$.

$2 + 2 + 3 + 3 = 10$

b) (i) Show that the set of all polynomials over $\mathbb{Z}$ with constant term zero is a prime ideal in $\mathbb{Z}[x]$ but not maximal there.

(ii) Define quotient space of a vector space.

Let $V = \mathbb{R}^4$ and W be a subspace of V generated by the vectors $(1,0,0,0), (1,1,0,1)$.

Find a basis of the quotient space V/W.

$(3 + 2) + (1 + 4) = 10$

***